

Minimal Nondeterministic Finite Automata and Atoms of Regular Languages [★]

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Abstract. We examine the NFA minimization problem in terms of atomic NFA's, that is, NFA's in which the right language of every state is a union of atoms, where the atoms of a regular language are non-empty intersections of complemented and uncomplemented left quotients of the language. We characterize all reduced atomic NFA's of a given language, that is, those NFA's that have no equivalent states. Using atomic NFA's, we formalize Sengoku's approach to NFA minimization and prove that his method fails to find all minimal NFA's. We also formulate the Kameda-Weiner NFA minimization in terms of quotients and atoms.

Keywords: regular language, quotient, atom, atomic NFA, minimal NFA

1 Introduction

Nondeterministic finite automata (NFA's) have played a major role in the theory of finite automata and regular expressions and their applications ever since their introduction in 1959 by Rabin and Scott [10]. In particular, the intriguing problem of finding NFA's with the minimal number of states has received much attention. The problem was first stated by Ott and Feinstein [8] in 1961. Various approaches have then been used over the years in attempts to answer this question; we mention a few examples here. In 1970, Kameda and Weiner [6] studied this problem using matrices related to the states of the minimal deterministic finite automata (DFA's) for a given language and its reverse. In 1992, Arnold, Dicky, and Nivat [1] used a "canonical" NFA. In the same year, Sengoku [11] used "normal" NFA's and "standard formed" NFA's. In 1995, Matz and Potthoff [7] returned to the "canonical" automaton and introduced the "fundamental" automaton. In 2003, Ilie and Yu [5] applied equivalence relations. In 2005, Polák [9] used the "universal" automaton.

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Our approach is to use the recently introduced atoms and atomic languages [3] for this question; we briefly state some of their basic properties here.

The *(left) quotient* of a regular language L over an alphabet Σ by a word $w \in \Sigma^*$ is the language $w^{-1}L = \{x \in \Sigma^* \mid wx \in L\}$. It is well known that the number of states in the complete minimal deterministic finite automaton recognizing L is precisely the number of distinct quotients of L . Also, L is its own quotient by the empty word ε , that is $\varepsilon^{-1}L = L$. A *quotient DFA* is a DFA uniquely determined by a regular language; its states correspond to left quotients. The quotient DFA is isomorphic to the minimal DFA.

An *atom*³ of a regular language L with quotients K_0, \dots, K_{n-1} is any non-empty language of the form $\widetilde{K_0} \cap \dots \cap \widetilde{K_{n-1}}$, where $\widetilde{K_i}$ is either K_i or $\overline{K_i}$, and $\overline{K_i}$ is the complement of K_i with respect to Σ^* . If the intersection with all quotients complemented is non-empty, then it constitutes the *negative* atom; all the other atoms are *positive*. Let the number of atoms be m , and let the number of positive atoms be p . Thus, if the negative atom is present, $p = m - 1$; otherwise, $p = m$.

So atoms of L are regular languages uniquely determined by L . They are pairwise disjoint and define a partition of Σ^* . Every quotient of L (including L itself) is a union of atoms, and every quotient of an atom is a union of atoms. Thus the atoms of a regular language are its basic building blocks. Also, \overline{L} defines the same atoms as L . The *átomaton* is an NFA uniquely determined by a regular language; its states correspond to atoms. An NFA is *atomic* if the right language of every state is a union of atoms.

Our contributions are as follows:

1. We characterize all trim reduced atomic NFA's of a given language, where an NFA is reduced if it has no equivalent states.
2. We show that, if n_0 is the minimal number of states of any NFA of a language, then the language may have trim reduced atomic NFA's with as few as n_0 states, and as many as $2^p - 1$ states.
3. We demonstrate that the number of atomic minimal NFA's can be as low as 1, or very high. For example, the language $\Sigma^*ab\Sigma^*$ with 3 quotients has 281 atomic minimal NFA's, and additional non-atomic ones.
4. We formalize the work of Sengoku [11] in our framework. He had no concept of atoms, but used an NFA equivalent to the átomaton and NFA's equivalent to atomic NFA's. Our use of atoms significantly clarifies Sengoku's method.
5. We prove that Sengoku's claim that an NFA can be made atomic by adding transitions and without changing the number of states is false. We show that there exist languages for which the minimal NFA's are all non-atomic. So Sengoku's claim that his method can always find a minimal NFA is also incorrect.
6. We formulate the Kameda-Weiner NFA minimization method [6] in terms of quotients and atoms.

³ The definition in [3] does not consider the intersection of all the complemented quotients to be an atom. Our new definition in [4] adds symmetry to the theory.

In Section 2 we recall some properties of automata and átomata. Atomic NFA's are then presented in Section 3. Sengoku's method is studied in Section 4, and the Kameda-Weiner method, in Section 5. Section 6 concludes the paper.

2 Automata and Átomata of Regular Languages

A *nondeterministic finite automaton (NFA)* is a quintuple $\mathfrak{N} = (Q, \Sigma, \eta, I, F)$, where Q is a finite, non-empty set of *states*, Σ is a finite non-empty *alphabet*, $\eta : Q \times \Sigma \rightarrow 2^Q$ is the *transition function*, $I \subseteq Q$ is the set of *initial states*, and $F \subseteq Q$ is the set of *final states*. As usual, we extend the transition function to functions $\eta' : Q \times \Sigma^* \rightarrow 2^Q$, and $\eta'' : 2^Q \times \Sigma^* \rightarrow 2^Q$, but use η for all three.

The *language accepted* by an NFA \mathfrak{N} is $L(\mathfrak{N}) = \{w \in \Sigma^* \mid \eta(I, w) \cap F \neq \emptyset\}$. Two NFA's are *equivalent* if they accept the same language. The *right language* of a state q is $L_{q,F}(\mathfrak{N}) = \{w \in \Sigma^* \mid \eta(q, w) \cap F \neq \emptyset\}$. The *right language* of a set S of states of \mathfrak{N} is $L_{S,F}(\mathfrak{N}) = \bigcup_{q \in S} L_{q,F}(\mathfrak{N})$; so $L(\mathfrak{N}) = L_{I,F}(\mathfrak{N})$. A state is *empty* if its right language is empty. Two states are *equivalent* if their right languages are equal. An NFA is *reduced* if it has no equivalent states. The *left language* of a state q is $L_{I,q} = \{w \in \Sigma^* \mid q \in \eta(I, w)\}$. A state is *unreachable* if its left language is empty. An NFA is *trim* if it has no empty or unreachable states. An NFA is *minimal* if it has the minimal number of states among all the equivalent NFA's.

A *deterministic finite automaton (DFA)* is a quintuple $\mathfrak{D} = (Q, \Sigma, \delta, q_0, F)$, where Q , Σ , and F are as in an NFA, $\delta : Q \times \Sigma \rightarrow Q$ is the transition function, and q_0 is the initial state.

We use the following operations on automata:

1. The *determinization* operation \mathbb{D} applied to an NFA \mathfrak{N} yields a DFA $\mathfrak{N}^{\mathbb{D}}$ obtained by the subset construction, where only subsets reachable from the initial subset of $\mathfrak{N}^{\mathbb{D}}$ are used, and the empty subset, if present, is included.
2. The *reversal* operation \mathbb{R} applied to NFA \mathfrak{N} yields an NFA $\mathfrak{N}^{\mathbb{R}}$, where the sets of initial and final states are interchanged and all transitions are reversed.
3. The *trimming* operation \mathbb{T} applied to an NFA deletes all unreachable and empty states.

The following theorem is from [2], and was also discussed in [3]:

Theorem 1 (Determinization). *If \mathfrak{D} is a DFA accepting a language L , then $\mathfrak{D}^{\mathbb{RD}}$ is a minimal DFA for L^R .*

Let L be any non-empty regular language, and let its set of quotients be $\mathcal{K} = \{K_0, \dots, K_{n-1}\}$. One of the quotients of L is L itself; this is called the *initial* quotient and is denoted by K_{in} . A quotient is *final* if it contains the empty word ε . The set of final quotients is $\mathcal{F} = \{K_i \mid \varepsilon \in K_i\}$.

In the following definition we use a 1-1 correspondence $K_i \leftrightarrow \mathbf{K}_i$ between quotients K_i of a language L and the states \mathbf{K}_i of the *quotient DFA* \mathfrak{D} defined below. We refer to the \mathbf{K}_i as *quotient symbols*.

Definition 1. The quotient DFA of L is $\mathfrak{D} = (\mathbf{K}, \Sigma, \delta, \mathbf{K}_{in}, \mathbf{F})$, where $\mathbf{K} = \{\mathbf{K}_0, \dots, \mathbf{K}_{n-1}\}$, \mathbf{K}_{in} corresponds to K_{in} , $\mathbf{F} = \{\mathbf{K}_i \mid K_i \in \mathcal{F}\}$, and $\delta(\mathbf{K}_i, a) = \mathbf{K}_j$ if and only if $a^{-1}K_i = K_j$, for all $\mathbf{K}_i, \mathbf{K}_j \in \mathbf{K}$ and $a \in \Sigma$.

In a quotient DFA the right language of \mathbf{K}_i is K_i , and its left language is $\{w \in \Sigma^* \mid w^{-1}L = K_i\}$. The language $L(\mathfrak{D})$ is the right language of \mathbf{K}_{in} , and hence $L(\mathfrak{D}) = L$. DFA \mathfrak{D} is minimal, since all quotients in K are distinct.

It follows from the definition of an atom, that a regular language L has at most 2^n atoms. An atom is *initial* if it has L (rather than \overline{L}) as a term; it is *final* if it contains ε . Since L is non-empty, it has at least one quotient containing ε . Hence it has exactly one final atom, the atom $\widehat{K_0} \cap \dots \cap \widehat{K_{n-1}}$, where $\widehat{K_i} = K_i$ if $\varepsilon \in K_i$, and $\widehat{K_i} = \overline{K_i}$ otherwise. Let $\mathcal{A} = \{A_0, \dots, A_{m-1}\}$ be the set of atoms of L . By convention, \mathcal{I} is the set of initial atoms, A_{p-1} is the final atom and the negative atom, if present, is A_{m-1} . The negative atom is not reachable from \mathcal{I} and can never be final, since there must be at least one final quotient in its intersection.

As above, we use a 1-1 correspondence $A_i \leftrightarrow \mathbf{A}_i$ between atoms A_i of a language L and the states \mathbf{A}_i of the NFA \mathfrak{A} defined below. We refer to the \mathbf{A}_i as *atom symbols*.

Definition 2. The átomaton of L is the NFA $\mathfrak{A} = (\mathbf{A}, \Sigma, \alpha, \mathbf{A}_I, \{\mathbf{A}_{p-1}\})$, where $\mathbf{A} = \{\mathbf{A}_i \mid A_i \in \mathcal{A}\}$, $\mathbf{A}_I = \{\mathbf{A}_i \mid A_i \in \mathcal{I}\}$, \mathbf{A}_{p-1} corresponds to A_{p-1} , and $\mathbf{A}_j \in \alpha(\mathbf{A}_i, a)$ if and only if $aA_j \subseteq A_i$, for all $\mathbf{A}_i, \mathbf{A}_j \in \mathbf{A}$ and $a \in \Sigma$.

In the átomaton, the right language of any state \mathbf{A}_i is the atom A_i .

The results from [3] and our definition of atoms in [4] imply that $\mathfrak{A}^{\mathbb{R}}$ is a minimal DFA that accepts L^R . It follows from Theorem 1 that $\mathfrak{A}^{\mathbb{R}}$ is isomorphic to $\mathfrak{D}^{\mathbb{RD}}$. The following result from [4] makes this isomorphism precise:

Theorem 2 (Isomorphism). Let \mathcal{S} be the collection of all subsets of the set K of quotient symbols. Let $\varphi : \mathbf{A} \rightarrow \mathcal{S}$ be the mapping assigning to state \mathbf{A}_j , corresponding to $A_j = K_{i_0} \cap \dots \cap K_{i_{n-r-1}} \cap \overline{K_{i_{n-r}}} \cap \dots \cap \overline{K_{i_{n-1}}}$ of $\mathfrak{A}^{\mathbb{R}}$, the set $\{\mathbf{K}_{i_0}, \dots, \mathbf{K}_{i_{n-r-1}}\}$. Then φ is a DFA isomorphism between $\mathfrak{A}^{\mathbb{R}}$ and $\mathfrak{D}^{\mathbb{RD}}$.

Corollary 1. The mapping φ is an NFA isomorphism between \mathfrak{A} and $\mathfrak{D}^{\mathbb{RDR}}$.

3 Atomic NFA's

A new class of NFA's was defined in [3] as follows:

Definition 3. An NFA $\mathfrak{N} = (Q, \Sigma, \eta, I, F)$ is atomic if for every $q \in Q$, the right language $L_{q,F}(\mathfrak{N})$ of q is a union of some positive atoms of $L(\mathfrak{N})$.

The following theorem, slightly restated, was proved in [3]:

Table 1. \mathfrak{N}_a .

		a	b
\rightarrow	0	$\{0, 1\}$	$\{0\}$
	1		$\{2\}$
\leftarrow	2	$\{2\}$	$\{2\}$

Table 2. \mathfrak{N}_b .

		a	b
\rightarrow	0	$\{1\}$	$\{0\}$
	1	$\{1\}$	$\{1, 2\}$
\leftarrow	2	$\{1, 2\}$	$\{0\}$

Table 3. \mathfrak{N}_c .

		a	b
\rightarrow	0	$\{1\}$	$\{0\}$
	1	$\{1\}$	$\{1, 2\}$
\leftarrow	2	$\{2\}$	

Theorem 3 (Atomicity). *A trim NFA \mathfrak{N} is atomic if and only if $\mathfrak{N}^{\mathbb{R}\mathbb{D}}$ is minimal.*

This theorem allows us to test whether an NFA \mathfrak{N} accepting a language L is atomic. To do this, reverse \mathfrak{N} and apply the subset construction. Then \mathfrak{N} is atomic if and only if $\mathfrak{N}^{\mathbb{R}\mathbb{D}}$ is isomorphic to the minimal DFA of L^R .

All three possibilities for the atomic nature of \mathfrak{N} and $\mathfrak{N}^{\mathbb{R}}$ exist: NFA \mathfrak{N}_a of Table 1 and its reverse are not atomic. NFA \mathfrak{N}_b of Table 2 is atomic, but its reverse is not. NFA \mathfrak{N}_c of Table 3 and its reverse are both atomic. Note that all three of these NFA's are equivalent, and they accept $\Sigma^*ab\Sigma^*$.

If we allow equivalent states, there is an infinite number of atomic NFA's, but their behaviours are not distinct; hence we consider only reduced NFA's. Suppose $\mathfrak{B} = (\mathcal{B}, \Sigma, \beta, \mathcal{B}_I, \mathcal{B}_F)$ is any trim reduced atomic NFA accepting L . Since \mathfrak{B} is atomic, the right language of any state in \mathfrak{B} is a union of positive atoms of L ; hence the states of \mathfrak{B} can be represented by sets of positive atom symbols. Because \mathfrak{B} is trim, it does not have a state with the empty set of atom symbols. Since \mathfrak{B} is reduced, no set of atom symbols appears twice. Thus the state set \mathcal{B} is a collection of non-empty sets of positive atom symbols.

Theorem 4 (Legality). *Suppose L is a regular language, its átomaton is $\mathfrak{A} = (\mathcal{A}, \Sigma, \alpha, \mathcal{A}_I, \{\mathcal{A}_{p-1}\})$, and $\mathfrak{B} = (\mathcal{B}, \Sigma, \beta, \mathcal{B}_I, \mathcal{B}_F)$ is a trim NFA, where $\mathcal{B} = \{\mathcal{B}_1, \dots, \mathcal{B}_r\}$ is a collection of sets of positive atom symbols and $\mathcal{B}_I, \mathcal{B}_F \subseteq \mathcal{B}$. If $\mathcal{B}_i \subseteq \mathcal{B}$, define $S(\mathcal{B}_i) = \bigcup_{\mathcal{B}_j \in \mathcal{B}_i} \mathcal{B}_j$ to be the set of atom symbols appearing in the sets \mathcal{B}_j of \mathcal{B}_i . Then \mathfrak{B} is a reduced atomic NFA of L if and only if it satisfies the following conditions:*

1. $S(\mathcal{B}_I) = \mathcal{A}_I$.
2. For all $\mathcal{B}_i \in \mathcal{B}$, $S(\beta(\mathcal{B}_i, a)) = \alpha(\mathcal{B}_i, a)$.
3. For all $\mathcal{B}_i \in \mathcal{B}$, we have $\mathcal{B}_i \in \mathcal{B}_F$ if and only if $\mathcal{A}_{p-1} \in \mathcal{B}_i$.

Before proving the theorem, we require the following lemma:

Lemma 1. *If \mathfrak{B} satisfies Condition 2 of Theorem 4, then $S(\beta(\mathcal{B}_i, w)) = \alpha(\mathcal{B}_i, w)$ for every $\mathcal{B}_i \in \mathcal{B}$ and $w \in \Sigma^*$.*

Proof. For $w = \varepsilon$, we have $S(\beta(\mathcal{B}_i, \varepsilon)) = S(\mathcal{B}_i) = \mathcal{B}_i$, and $\alpha(\mathcal{B}_i, \varepsilon) = \mathcal{B}_i$; so the claim holds for this case.

Assume that $S(\beta(\mathcal{B}_i, w)) = \alpha(\mathcal{B}_i, w)$ for all $\mathcal{B}_i \in \mathcal{B}$ and all $w \in \Sigma^*$ with length less than or equal to $l \geq 0$. We prove that $S(\beta(\mathcal{B}_i, wa)) = \alpha(\mathcal{B}_i, wa)$ for

every $a \in \Sigma$. Let $\beta(\mathbf{B}_i, w) = \{\mathbf{B}_{i_1}, \dots, \mathbf{B}_{i_h}\}$ for some $\mathbf{B}_{i_1}, \dots, \mathbf{B}_{i_h} \in \mathcal{B}$. Since $\beta(\mathbf{B}_i, wa) = \beta(\beta(\mathbf{B}_i, w), a) = \beta(\mathbf{B}_{i_1}, a) \cup \dots \cup \beta(\mathbf{B}_{i_h}, a)$, we have $S(\beta(\mathbf{B}_i, wa)) = S(\beta(\mathbf{B}_{i_1}, a) \cup \dots \cup \beta(\mathbf{B}_{i_h}, a)) = S(\beta(\mathbf{B}_{i_1}, a)) \cup \dots \cup S(\beta(\mathbf{B}_{i_h}, a))$. By Condition 2, the latter is equal to $\alpha(\mathbf{B}_{i_1}, a) \cup \dots \cup \alpha(\mathbf{B}_{i_h}, a) = \alpha(\mathbf{B}_{i_1} \cup \dots \cup \mathbf{B}_{i_h}, a) = \alpha(S(\beta(\mathbf{B}_i, w)), a)$. By the inductive assumption, we get $\alpha(S(\beta(\mathbf{B}_i, w)), a) = \alpha(\alpha(\mathbf{B}_i, w), a) = \alpha(\mathbf{B}_i, wa)$, which proves our claim. \square

Proof of Theorem 4

Proof. First we prove that any NFA \mathfrak{B} satisfying Conditions 1–3 is an atomic NFA of L . Let $\mathbf{B}_i \in \mathcal{B}$ be a state of \mathfrak{B} . If $w \in L_{\mathbf{B}_i, \mathcal{B}_F}(\mathfrak{B})$, then by Condition 3, there exists $\mathbf{B}_j \in \beta(\mathbf{B}_i, w)$ such that $\mathbf{A}_{p-1} \in \mathbf{B}_j$, and we have $\mathbf{A}_{p-1} \in S(\beta(\mathbf{B}_i, w))$. By Lemma 1, we get $\mathbf{A}_{p-1} \in \alpha(\mathbf{B}_i, w)$, implying that there is some $\mathbf{A}_k \in \mathbf{B}_i$ such that $w \in L_{\mathbf{A}_k, \{\mathbf{A}_{p-1}\}}(\mathfrak{A})$. Conversely, if $w \in L_{\mathbf{A}_k, \{\mathbf{A}_{p-1}\}}(\mathfrak{A})$ and $\mathbf{A}_k \in \mathbf{B}_i$, then $\mathbf{A}_{p-1} \in \alpha(\mathbf{B}_i, w) = S(\beta(\mathbf{B}_i, w))$. Hence there exists $\mathbf{B}_j \in \beta(\mathbf{B}_i, w)$ such that $\mathbf{A}_{p-1} \in \mathbf{B}_j$. Consequently, every word accepted in \mathfrak{B} from state \mathbf{B}_i is in some atom A_k such that $\mathbf{A}_k \in \mathbf{B}_i$, and every word in an atom A_k such that $\mathbf{A}_k \in \mathbf{B}_i$, is also in $L_{\mathbf{B}_i, \mathcal{B}_F}(\mathfrak{B})$. Therefore the right language of \mathbf{B}_i in \mathfrak{B} is equal to the union of atoms A_k such that $\mathbf{A}_k \in \mathbf{B}_i$. In particular, $L_{\mathbf{B}_i, \mathcal{B}_F}(\mathfrak{B})$ is the union of atoms whose atom symbols appear in the initial collection of \mathfrak{B} which, by Condition 1, is the same as the union of atoms whose atom symbols are initial in \mathfrak{A} . But that last union is precisely $L_{\mathbf{A}_I, \{\mathbf{A}_{p-1}\}}(\mathfrak{A}) = L$. Since any two sets \mathbf{B}_i and \mathbf{B}_j are different, and atoms are disjoint, \mathfrak{B} is reduced. Hence \mathfrak{B} is a reduced atomic NFA of L .

Conversely, we show that if \mathfrak{B} is a reduced atomic NFA of L , then it must satisfy Conditions 1–3. So in the following we assume that \mathfrak{B} is atomic, that is, for every state \mathbf{B}_i of \mathfrak{B} , the right language of \mathbf{B}_i is equal to the union of atoms A_k such that $\mathbf{A}_k \in \mathbf{B}_i$.

First, we show that Condition 1 holds. Let $\mathbf{A}_j \in S(\mathcal{B}_I)$. Then there is a state $\mathbf{B}_j \in \mathcal{B}_I$ such that $\mathbf{A}_j \in \mathbf{B}_j$. So for any $w \in A_j$, $w \in L(\mathfrak{B})$. Since $L(\mathfrak{B}) = L(\mathfrak{A})$, we have $w \in L(\mathfrak{A})$ for all $w \in A_j$. Thus $\mathbf{A}_j \in \mathbf{A}_I$. Conversely, if $\mathbf{A}_j \in \mathbf{A}_I$, then for all $w \in A_j$, $w \in L(\mathfrak{A}) = L(\mathfrak{B})$. Since \mathfrak{B} is atomic, there is an initial state \mathbf{B}_j such that $A_j \subseteq L_{\mathbf{B}_j, \mathcal{B}_F}(\mathfrak{B})$. Hence $\mathbf{A}_j \in S(\mathcal{B}_I)$.

Next, we prove Condition 2. If $\mathbf{A}_j \in S(\beta(\mathbf{B}_i, a))$, then $L_{\mathbf{B}_i, \mathcal{B}_F}(\mathfrak{B})$ must contain aA_j . So there must exist some $\mathbf{A}_i \in \mathbf{B}_i$ such that $aA_j \subseteq A_i$. Thus $\mathbf{A}_j \in \alpha(\mathbf{B}_i, a)$. Conversely, if $\mathbf{A}_j \in \alpha(\mathbf{B}_i, a)$, then there is an atom $\mathbf{A}_i \in \mathbf{B}_i$ such that $\mathbf{A}_j \in \alpha(\mathbf{A}_i, a)$, implying $aA_j \subseteq A_i$. Since $\mathbf{A}_i \in \mathbf{B}_i$, $L_{\mathbf{B}_i, \mathcal{B}_F}(\mathfrak{B})$ must contain aA_j . Hence $\mathbf{A}_j \in S(\beta(\mathbf{B}_i, a))$.

To show that Condition 3 holds, we first suppose that $\mathbf{B}_i \in \mathcal{B}_F$. Then ε is in the right language of \mathbf{B}_i . Since \mathfrak{B} is atomic, ε must be in one of the atoms of \mathbf{B}_i . However, the only atom containing ε is A_{p-1} , so $\mathbf{A}_{p-1} \in \mathbf{B}_i$. Conversely, if $\mathbf{A}_{p-1} \in \mathbf{B}_i$, then ε is in the right language of \mathbf{B}_i , and \mathbf{B}_i is a final state by definition of an NFA. \square

Example 1. Consider the trim átomaton \mathfrak{A}^T of Table 4 and the atomic NFA \mathfrak{B} of Table 5. Here $\mathcal{B} = \{\mathbf{B}_0, \mathbf{B}_1, \mathbf{B}_2\}$, where $\mathbf{B}_0 = \{\mathbf{A}_0, \mathbf{A}_1\}$, $\mathbf{B}_1 = \{\mathbf{A}_2\}$, and $\mathbf{B}_2 = \{\mathbf{A}_0, \mathbf{A}_2\}$. The initial collection is $\mathcal{B}_I = \{\mathbf{B}_0\} = \{\{\mathbf{A}_0, \mathbf{A}_1\}\}$, and the

Table 4. Átomaton $\mathfrak{A}^\mathbb{T}$.

		a	b
\rightarrow	\mathbf{A}_0	$\{\mathbf{A}_0, \mathbf{A}_1\}$	$\{\mathbf{A}_0, \mathbf{A}_2\}$
\rightarrow	\mathbf{A}_1	$\{\mathbf{A}_2\}$	
\leftarrow	\mathbf{A}_2		

Table 5. Atomic NFA \mathfrak{B} .

		a	b
\rightarrow	$\{\mathbf{A}_0, \mathbf{A}_1\}$	$\{\{\mathbf{A}_0, \mathbf{A}_1\}, \{\mathbf{A}_2\}\}$	$\{\{\mathbf{A}_0, \mathbf{A}_2\}\}$
\leftarrow	$\{\mathbf{A}_2\}$		
\leftarrow	$\{\mathbf{A}_0, \mathbf{A}_2\}$	$\{\{\mathbf{A}_0, \mathbf{A}_1\}\}$	$\{\{\mathbf{A}_0, \mathbf{A}_2\}\}$

final collection is $\mathcal{B}_F = \{\mathcal{B}_1, \mathcal{B}_2\} = \{\{\mathbf{A}_2\}, \{\mathbf{A}_0, \mathbf{A}_2\}\}$. One verifies that all the conditions of Theorem 4 hold, and NFA's $\mathfrak{A}^\mathbb{T}$ and \mathfrak{B} are equivalent. \blacksquare

The number of trim reduced atomic NFA's can be very large. There can be such NFA's with as many as $2^p - 1$ non-empty states, since there are that many non-empty sets of positive atoms. However, in a general case, not all sets of positive atom symbols can be states of an atomic NFA. The largest reduced atomic NFA is characterized in the following theorem.

Theorem 5 (Maximal atomic NFA). *If \mathcal{B} is the collection of all sets \mathcal{B}_i such that \mathcal{B}_i is a non-empty subset of the set of positive atom symbols $\{\mathbf{A}_h \mid A_h \subseteq K_j\}$ of any quotient K_j of L , then there exists a trim reduced atomic NFA of L with state set \mathcal{B} .*

Proof. Let $\mathfrak{B} = (\mathcal{B}, \Sigma, \beta, \mathcal{B}_I, \mathcal{B}_F)$ be an NFA in which the state set \mathcal{B} is the collection of all sets \mathcal{B}_i such that \mathcal{B}_i is a non-empty subset of the set of atom symbols $\{\mathbf{A}_h \mid A_h \subseteq K_j\}$ of any quotient K_j of L , where $j \in \{0, \dots, n-1\}$, $\beta(\mathcal{B}_i, a) = \{\mathcal{B}_j \mid \mathcal{B}_j \subseteq \alpha(\mathcal{B}_i, a)\}$ for every $\mathcal{B}_i \in \mathcal{B}$ and $a \in \Sigma$, $\mathcal{B}_i \in \mathcal{B}_I$ if and only if \mathcal{B}_i is a subset of the set of atom symbols of the initial quotient K_{in} , and $\mathcal{B}_i \in \mathcal{B}_F$ if and only if $\mathbf{A}_{p-1} \in \mathcal{B}_i$. We claim that \mathfrak{B} is a trim reduced atomic NFA of L .

First, we show that \mathfrak{B} is trim. Let us consider any state \mathcal{B}_i of \mathfrak{B} . Let K_j be a quotient such that \mathcal{B}_i is a subset of the set of atom symbols of K_j , and let \mathcal{B}_j be the set of atom symbols corresponding to K_j . Let \mathcal{B}_0 be the set of atom symbols corresponding to the initial quotient K_{in} of L . Note that $\mathcal{B}_0 = \mathbf{A}_I$. Since every set of atom symbols corresponding to some quotient is reachable from the initial set of atom symbols in the átomaton \mathfrak{A} , there must be a word $w \in \Sigma^*$, such that \mathcal{B}_j is reachable from \mathcal{B}_0 by w in \mathfrak{A} . We show that \mathcal{B}_i is reachable from some initial state of \mathfrak{B} by w . If $w = \varepsilon$, then $K_j = K_{in}$, and since $\mathcal{B}_i \subseteq \mathcal{B}_j$, it follows that \mathcal{B}_i is an initial state of \mathfrak{B} reachable from itself by ε . If $w = ua$ for some $u \in \Sigma^*$ and $a \in \Sigma$, then there is a state \mathcal{B}_u of \mathfrak{B} , reachable from \mathcal{B}_0 by u , such that \mathcal{B}_u corresponds to the quotient $u^{-1}L$ of L and $\mathcal{B}_j = \alpha(\mathcal{B}_u, a)$. Since $\mathcal{B}_i \subseteq \mathcal{B}_j$ and $\mathcal{B}_j = \alpha(\mathcal{B}_u, a)$, by the definition of β we have $\mathcal{B}_i \in \beta(\mathcal{B}_u, a)$. Thus, \mathcal{B}_i is reachable from \mathcal{B}_0 in \mathfrak{B} by ua .

We also have to show that there is a word $w \in \Sigma^*$, such that some final state of \mathfrak{B} is reachable from \mathcal{B}_i by w . If \mathcal{B}_i is final, then it is reachable from itself by $w = \varepsilon$. If \mathcal{B}_i is not final, then let us consider any $\mathbf{A}_k \in \mathcal{B}_i$. Since the right language of the state \mathbf{A}_k in the átomaton \mathfrak{A} is not empty, and \mathbf{A}_k cannot be the final state of \mathfrak{A} , there must be some state \mathbf{A}_l of \mathfrak{A} and some $a \in \Sigma$, such

that $\mathbf{A}_l \in \alpha(\mathbf{A}_k, a)$. Now we know that there is some \mathbf{B}_j such that $\mathbf{A}_l \in \mathbf{B}_j$ and $\alpha(\mathbf{B}_i, a) = \mathbf{B}_j$. Since $\beta(\mathbf{B}_i, a)$ is the collection of all non-empty subsets of \mathbf{B}_j , it follows that $\{\mathbf{A}_l\} \in \beta(\mathbf{B}_i, a)$. Since the final state \mathbf{A}_{p-1} of \mathfrak{A} is reachable from \mathbf{A}_l by any word $v \in A_l$, we get $\{\mathbf{A}_{p-1}\} \in \beta(\mathbf{B}_i, av)$ by the definition of β . So a final state $\{\mathbf{A}_{p-1}\}$ of \mathfrak{B} is reachable from \mathbf{B}_i by av . Thus, \mathfrak{B} is trim.

To see that \mathfrak{B} is a reduced atomic NFA, one verifies that Conditions 1–3 of Theorem 4 hold. Thus by Theorem 4, \mathfrak{B} is a trim reduced atomic NFA of L . \square

Theorem 6 (NFA with $2^p - 1$ states). *A regular language L has a trim reduced atomic NFA with $2^p - 1$ states if and only if for some quotient K_i of L , $K_i = A_0 \cup \dots \cup A_{p-1}$.*

Proof. Let $\mathfrak{B} = (\mathcal{B}, \Sigma, \beta, \mathcal{B}_I, \mathcal{B}_F)$ be a trim reduced atomic NFA of L with $2^p - 1$ states. Then there must be a state \mathbf{B}_i of \mathfrak{B} such that $\mathbf{B}_i = \{\mathbf{A}_0, \dots, \mathbf{A}_{p-1}\}$. Since the right language of any state of a trim NFA is a subset of some quotient, we have $L_{\mathbf{B}_i, \mathcal{B}_F}(\mathfrak{B}) = A_0 \cup \dots \cup A_{p-1} \subseteq K_i$ for some quotient K_i of L . On the other hand, K_i must be a union of some positive atoms, so we get $K_i = A_0 \cup \dots \cup A_{p-1}$.

Conversely, let $K_i = A_0 \cup \dots \cup A_{p-1}$ be a quotient of L which includes all the positive atoms of L . Then by Theorem 5, there is a trim reduced atomic NFA of L in which the state set is the collection of all non-empty subsets of the set of positive atom symbols. This NFA has $2^p - 1$ states. \square

The construction of reduced atomic NFA's is illustrated in the following example. To simplify the notation, we do not use atom symbols in examples.

Example 2. Consider the minimal DFA \mathfrak{D} taken from [6] and shown in Table 6. It accepts the language $L = \Sigma^*(b \cup aa) \cup a$, and its quotients are $K_0 = \varepsilon^{-1}L = L$, $K_1 = a^{-1}L = \Sigma^*(b \cup aa) \cup a \cup \varepsilon$, and $K_2 = b^{-1}L = \Sigma^*(b \cup aa) \cup \varepsilon$. NFA $\mathfrak{D}^{\text{RDRT}}$ and the isomorphic trim átomaton \mathfrak{A}^{T} with states renamed are shown in Tables 7 and 8. The positive atoms are $A = \Sigma^*(b \cup aa)$, $B = a$ and $C = \varepsilon$, and $K_0 = A \cup B$, $K_1 = A \cup B \cup C$, and $K_2 = A \cup C$.

Since the set $\{A, B\}$ of initial atoms does not contain all positive atoms, no 1-state NFA exists.

1. For the initial state we could pick one state $\{A, B\}$ with two atoms. From there, the átomaton reaches $\{A, B, C\}$ under a , and $\{A, C\}$ under b .
 - (a) If we pick $\{A, C\}$ as the second state, we can cover $\{A, B, C\}$ by $\{A, B\}$ and $\{A, C\}$, as in Table 9. Here the minimal atomic NFA is unique.

Table 6. \mathfrak{D} .

		a	b
\rightarrow	0	1	2
\leftarrow	1	1	2
\leftarrow	2	0	2

Table 7. $\mathfrak{D}^{\text{RDRT}}$.

		a	b
\leftarrow	12		
\rightarrow	01	{12}	
\rightarrow	012	{012, 01}	{012, 12}

Table 8. \mathfrak{A}^{T} .

		a	b
\leftarrow	C		
\rightarrow	B	{ C }	
\rightarrow	A	{ A, B }	{ A, C }

Table 9. NFA \mathfrak{B}_1 .

		a	b
\rightarrow	$\{A, B\}$	$\{A, B\}, \{A, C\}$	$\{A, C\}$
\leftarrow	$\{A, C\}$	$\{A, B\}$	$\{A, C\}$

Table 10. Atomic NFA \mathfrak{B}_2 .

		a	b
\rightarrow	$\{A, B\}$	$\{A, B\}, \{C\}$	$\{A, C\}$
\leftarrow	$\{C\}$		
\leftarrow	$\{A, C\}$	$\{A, B\}$	$\{A, C\}$

Table 12. A 7-state NFA.

Table 11. A 5-state NFA.

		a	b
\rightarrow	$\{A\}$	$\{A\}, \{B\}$	$\{A, C\}$
\rightarrow	$\{B\}$	$\{C\}$	
\leftarrow	$\{A, C\}$	$\{A, B\}$	$\{A, C\}$
\leftarrow	$\{C\}$		
	$\{A, B\}$	$\{A, B\}, \{C\}$	$\{A\}, \{C\}$

		a	b
\rightarrow	$\{A\}$	$\{A\}, \{B\}$	$\{A, C\}$
\rightarrow	$\{B\}$	$\{C\}$	
\leftarrow	$\{A, C\}$	$\{A, B\}$	$\{A, C\}$
\leftarrow	$\{C\}$		
\rightarrow	$\{A, B\}$	$\{A, B, C\}, \{B, C\}$	$\{A, C\}$
\leftarrow	$\{A, B, C\}$	$\{A, B, C\}, \{B, C\}$	$\{A, C\}$
\leftarrow	$\{B, C\}$	$\{C\}$	

- (b) We can also use $\{A, B, C\}$ as a state. Then we need $\{A, C\}$ for the transition under b . This gives an NFA isomorphic to the DFA of Table 6.
- (c) We can use state $\{C\}$ as shown in Table 10.
- 2. We can pick two initial states, $\{A\}$ and $\{B\}$.
 - (a) If we add $\{C\}$, this leads to the átomaton of Table 8.
 - (b) A 5-state solution is shown in Table 11.
- 3. We can use three initial states, $\{A\}$, $\{B\}$ and $\{A, B\}$. A 7-state NFA is shown in Table 12. This is a largest possible reduced solution. ■

The number of minimal atomic NFA's can also be very large.

Example 3. Let $\Sigma = \{a, b\}$ and consider the language $L = \Sigma^* a \Sigma^* b \Sigma^* = \Sigma^* ab \Sigma^*$. The quotients of L are $K_0 = L$, $K_1 = L \cup b \Sigma^*$ and $K_2 = \Sigma^*$. The quotient DFA of L is shown in Table 13, and its átomaton, in Tables 14 and 15 (where the atoms have been relabelled). The atoms are $A = L$, $B = b^* b a^*$ and $C = a^*$, and there is no negative atom. Thus the quotients are $K_0 = L = A$, $K_1 = A \cup B$, and $K_2 = A \cup B \cup C$.

We find all the minimal atomic NFA's of L . Obviously, there is no 1-state solution. The states of any atomic NFA are sets of atoms, and there are seven non-empty sets of atoms to choose from. Since there is only one initial atom, there is no choice: we must take $\{A\}$. For the transition $(A, a, \{A, B\})$, we can add $\{B\}$ or $\{A, B\}$. If there are only two states, atom $\{C\}$ cannot be reached. So there is no 2-state atomic NFA. The results for 3-state atomic NFA's are summarized in Proposition 1.

Proposition 1. *The language $\Sigma^* ab \Sigma^*$ has 281 minimal atomic NFA's.*

Table 13. DFA \mathfrak{D} .

		a	b
\rightarrow	0	1	0
	1	1	2
\leftarrow	2	2	2

Table 14. Átomaton \mathfrak{A} .

		a	b
\leftarrow	2	$\{2\}$	
	12		$\{12, 2\}$
\rightarrow	012	$\{012, 12\}$	$\{012\}$

Table 15. \mathfrak{A} relabelled.

		a	b
\leftarrow	C	$\{C\}$	
	B		$\{B, C\}$
\rightarrow	A	$\{A, B\}$	$\{A\}$

Table 16. NFA \mathfrak{N}_2 .

		a	b
\rightarrow	A	AB	A
	AB	AB	AB, C
\leftarrow	C	C	

Table 17. NFA \mathfrak{N}_9 .

		a	b
\rightarrow	A	A, AB	A
	AB	A, AB	A, AB, C
\leftarrow	C	C	

Proof. We concentrate on 3-state solutions. We drop the curly brackets and commas and represent sets of atoms by words. Thus $\{A, AB, BC\}$ stands for $\{\{A\}, \{A, B\}, \{B, C\}\}$.

State A is the only initial state and so it must be included. To implement the transition $(A, a, \{A, B\})$ from \mathfrak{A} , either B or AB must be chosen.

1. If B is chosen, then there must be a set containing C but not A ; otherwise the transition $(B, b, \{B, C\})$ cannot be realized.
 - (a) If BC is taken, then C must be taken, and this would make four states.
 - (b) Hence C must be chosen, giving states A , B , and C . This yields the átomaton $\mathfrak{A} = \mathfrak{N}_1$.
2. If AB is chosen, then we could choose C , AC or ABC , since BC would also require C . Thus there are three cases:
 - (a) $\{A, AB, C\}$ yields \mathfrak{N}_2 of Table 16, if the minimal number of transitions is used. The following transitions can also be added: (A, a, A) , (AB, a, A) , (AB, b, A) . Since these can be added independently, we have eight more NFA's. Using the maximal number of transitions, we get \mathfrak{N}_9 of Table 17.
 - (b) $\{A, AB, AC\}$ results in \mathfrak{N}_{10} with the minimal number of transitions, and \mathfrak{N}_{25} with the maximal one.
 - (c) $\{A, AB, ABC\}$ results in \mathfrak{N}_{26} (the quotient DFA) with the minimal number of transitions, and \mathfrak{N}_{281} with the maximal one.

Table 18. NFA \mathfrak{N}_{10} .

		a	b
\rightarrow	A	AB	A
	AB	AB	AB, AC
\leftarrow	AC	AB, AC	A

Table 19. NFA \mathfrak{N}_{25} .

		a	b
\rightarrow	A	A, AB	A
	AB	A, AB	A, AB, AC
\leftarrow	AC	A, AB, AC	A

Table 20. NFA \mathfrak{N}_{26} .

		a	b
\rightarrow	A	AB	A
	AB	AB	ABC
\leftarrow	ABC	ABC	ABC

Table 21. NFA \mathfrak{N}_{281} .

		a	b
\rightarrow	A	A, AB	A
	AB	A, AB	A, AB, ABC
\leftarrow	ABC	A, AB, ABC	A, AB, ABC

Table 22. NFA \mathfrak{N}_{282} .

		a	b
\rightarrow	0	1	0
	1	1	0, 1, 2
\leftarrow	2	0, 2	

As well, L has 3-state non-atomic NFA's. The determinized version of NFA \mathfrak{N}_{10} of Table 18 is not minimal. By Theorem 3, $\mathfrak{N}_{10}^{\mathbb{R}}$ is not atomic. But $L^R = \Sigma^*ba\Sigma^*$; hence we obtain a non-atomic 3-state NFA for L by reversing \mathfrak{N}_{10} and interchanging a and b . That NFA with renamed states is shown in Table 22.

The right languages of the states of \mathfrak{N}_{282} are: $L_0 = L = A$, $L_1 = A \cup B$, and $L_2 = \varepsilon \cup a \cup aa\Sigma^* \cup abb^*aa^*b\Sigma^*$, which is not a union of atoms. Six more non-atomic NFA's can be derived from NFA's between \mathfrak{N}_{10} and \mathfrak{N}_{25} . \square

This is a rather large number of NFA's for a language with 3 quotients. \blacksquare

One can verify that there is no NFA with fewer than 3 states which accepts the language $L = \Sigma^*ab\Sigma^*$. This implies that every minimal atomic NFA of L is also a minimal NFA of L . However, this is not the case with all regular languages, as we will see in the next section.

4 Sengoku's NFA Minimization Method

Sengoku had no concept of atom, but he came very close to discovering it. For a language accepted by a minimal DFA \mathfrak{D} , the *normal* NFA [11](p. 18) is isomorphic to $\mathfrak{D}^{\text{RDRT}}$, and hence to the trim átomaton, by our Corollary 1. Moreover, he defines an NFA \mathfrak{N} to be in *standard form* [11](p. 19) if $\mathfrak{N}^{\text{RID}}$ is minimal. By our Theorem 3, such an \mathfrak{N} is atomic. Sengoku makes the following claim [11](p. 20):

We can transform the nondeterministic automaton into its standard form by adding some extra transitions to the automaton. Therefore the number of states is unchangeable.

This claim amounts to stating that any NFA can be transformed to an equivalent atomic NFA by adding some transitions. Unfortunately, the claim is false:

Theorem 7. *There exists a language for which no minimal NFA is atomic.*

Proof. This example is from [7]. A quotient DFA \mathfrak{D} , the NFA $\mathfrak{D}^{\text{RDR}}$, and its isomorphic átomaton \mathfrak{A} with relabelled states are in Tables 23–25, respectively (there is no negative atom). We now drop the curly brackets and commas in tables, and represent sets of atoms by words. A minimal NFA \mathfrak{N}_{\min} of this language, having four states, is shown in Table 26; it is not atomic and it is not unique. We try to construct a 4-state atomic NFA $\mathfrak{N}_{\text{atom}}$ equivalent to \mathfrak{D} .

Table 23. \mathfrak{D} .

		a	b
\rightarrow	0	1	2
	1	3	4
\leftarrow	2	5	4
	3	3	1
	4	6	2
\leftarrow	5	7	2
	6	3	8
\leftarrow	7	7	7
	8	6	7

Table 24. $\mathfrak{D}^{\text{RDR}}$.

		a	b
\leftarrow	257	257, 04578	
\rightarrow	04578	12678	257
	12678		04578, 03 – 8
\rightarrow	03 – 8		12678
	1 – 8	03 – 8	
\rightarrow	0 – 8	1 – 8, 0 – 8	1 – 8, 0 – 8

Table 25. \mathfrak{A} .

		a	b
\leftarrow	A	AB	
\rightarrow	B	C	A
	C		BD
\rightarrow	D		C
	E	D	
\rightarrow	F	EF	EF

First, we note that quotients corresponding to the states of \mathfrak{D} can be expressed as sets of atoms as follows: $K_0 = \{B, D, F\}$, $K_1 = \{C, E, F\}$, $K_2 = \{A, C, E, F\}$, $K_3 = \{D, E, F\}$, $K_4 = \{B, D, E, F\}$, $K_5 = \{A, B, D, E, F\}$, $K_6 = \{C, D, E, F\}$, $K_7 = \{A, B, C, D, E, F\}$, and $K_8 = \{B, C, D, E, F\}$. One can verify that these are the states of the determinized version of the átomaton, which is isomorphic to the original DFA \mathfrak{D} . Now, every state of $\mathfrak{N}_{\text{atom}}$ must be a subset of a set of atoms of some quotient, and all these sets of atoms of quotients must be covered by the states of $\mathfrak{N}_{\text{atom}}$. We note that quotients $\{B, D, F\}$, $\{C, E, F\}$, and $\{D, E, F\}$ do not contain any other quotients as subsets, while all the other quotients do. It is easy to see that there is no combination of three or fewer sets of atoms, other than these three sets, that can cover these quotients. So we have to use these sets as states of $\mathfrak{N}_{\text{atom}}$. We also need at least one set containing the atom A . If we use only one set of atoms with A , that set has to be a subset of every quotient having A . So it must be a subset of $\{A, E, F\}$. If we use $\{A\}$ as a state, then by the transition table of the átomaton, there must be at least one more state to cover $\{A, B\}$. Similarly, if we use $\{A, E\}$, then we must have another state to cover $\{A, B, D\}$. If we use $\{A, F\}$, then we must have a state to cover $\{A, B, E, F\}$. And if we use $\{A, E, F\}$, then we must have a state to cover $\{E, F\}$. We conclude that a smallest atomic NFA has at least five states. There is a five-state atomic NFA, as shown in Table 27. It is not unique.

Since there does not exist a four-state atomic NFA equivalent to the DFA \mathfrak{D} , it is not possible to convert the non-atomic minimal NFA \mathfrak{N}_{\min} to an atomic NFA by adding transitions. \square

In summary, Sengoku’s method cannot find the minimal NFA’s in all cases. However, it is able to find all atomic minimal NFA’s. His minimization algorithm

Table 27. \mathfrak{N}_{atom} .Table 26. NFA \mathfrak{N}_{min} .

		a	b
\rightarrow	0	1	1, 2
	1	3	0, 3
\leftarrow	2	0, 2, 3	
	3	3	1

		a	b
\rightarrow	BDF	CEF	CEF, AEF
	CEF	DEF	BDF, DEF
\leftarrow	AEF	BDF, AEF, DEF	EF
	DEF	DEF	CEF
	EF	DEF	EF

proceeds by “merging some states of the normal nondeterministic automaton.” This is similar to our search for subsets of atoms that satisfy Theorem 4.

5 The Kameda-Weiner Minimization Method

We present a short and modified outline of the properties of the Kameda-Weiner NFA minimization method [6] using mostly our terminology and notation. They consider a trim minimal DFA $\mathfrak{D} = (Q, \Sigma, \delta, q_0, F)$ with Q of cardinality n , and its reversed determinized and trim version $\mathfrak{D}^{\text{RDT}}$; the set of states of $\mathfrak{D}^{\text{RDT}}$ is a subset \mathcal{S} of cardinality p of $2^Q \setminus \emptyset$. They then form an $n \times p$ matrix T where the rows correspond to non-empty states $q_i \in Q$ of \mathfrak{D} , which is the trim minimal DFA of a language L , and columns, to states $S_j \in \mathcal{S}$ of $\mathfrak{D}^{\text{RDT}}$, which is the trim minimal DFA of the language L^R by Theorem 1. The entry $t_{i,j}$ of the matrix T is 1 if $q_i \in S_j$, and 0 otherwise.

We use $\mathfrak{D}^{\text{RDT}}$, the trim átomaton, instead of $\mathfrak{D}^{\text{RDT}}$, since the state sets of these two automata are identical. Interpret the rows of the matrix as non-empty quotients of L and columns, as positive atoms of L . Then $t_{i,j} = 1$ if and only if quotient K_i contains atom A_j , and it is clear that every regular language defines a unique such matrix, which we will refer to as the *quotient-atom matrix*.

The ordered pair (K_i, A_j) with $K_i \in \mathcal{K}$ and $A_i \in \mathcal{A}$ is a *point* of T if $t_{i,j} = 1$. A *grid* g of T is the direct product $g = P \times R$ of a set P of quotients with a set R of atoms. If $g = P \times R$ and $g' = P' \times R'$ are two grids of T , then $g \subseteq g'$ if and only if $P \subseteq P'$ and $R \subseteq R'$. Thus \subseteq is a partial order on the set of all grids of T , and a grid is *maximal* if it is not contained in any other grid. A *cover* C of T is a set $C = \{g_0, \dots, g_{k-1}\}$ of grids, such that every point (K_i, A_j) belongs to some grid g_i in C . A *minimal cover* has the minimal number of grids.

Let $f : \mathcal{K} \rightarrow 2^C \setminus \emptyset$ be the function that assigns to quotient $K_i \in \mathcal{K}$ the set of grids $g = P \times R$ such that $K_i \in P$. The NFA constructed by the Kameda-Weiner method is $\mathfrak{N}_C = (C, \Sigma, \eta_C, C_I, C_F)$, where C is a cover consisting of maximal grids, $C_I = f(K_{in})$ is the set of grids corresponding to the initial quotient K_{in} , and C_F is defined by $g \in C_F$ if and only if $g \in f(K_i)$ implies that K_i is a final quotient. For every grid $g = P \times R$ and $x \in \Sigma$, we can compute $\eta_C(g, x)$ by the formula $\eta_C(g, x) = \bigcap_{K_i \in P} f(x^{-1}K_i)$.

It may be the case that \mathfrak{N}_C is not equivalent to DFA \mathfrak{D} . A cover C is called *legal* if $L(\mathfrak{N}_C) = L(\mathfrak{D})$. To find a minimal NFA of a language L , the method

in [6] tests the covers of the quotient-atom matrix of L in the order of increasing size to see if they are legal. The first legal NFA is a minimal one.

When we apply the Kameda-Weiner method [6] to the example in Theorem 7, we get the NFA of Table 26.

We apply the Kameda-Weiner method [6] to the example in Theorem 7. The quotients in the example are referred to as the integers 0–8, as in Table 23. The atoms are those in Table 24 relabelled as in Table 25. The quotient-atom matrix is shown in Table 28, where the non-blank entries are to be interpreted as 1's and the blank entries as 0's. Table 28 also shows a minimal cover $S = (g_0, g_1, g_2, g_3)$ and $f(K_i)$ for each quotient K_i of \mathcal{K} .

Table 28. Cover C for quotient-atom matrix of \mathfrak{D} .

		F	E	D	C	B	A	$f(K_i)$
\rightarrow	0	g_0		g_0		g_0		$\{g_0\}$
	1	g_1	g_1		g_1			$\{g_1\}$
\leftarrow	2	g_1, g_2	g_1, g_2		g_1		g_2	$\{g_1, g_2\}$
	3	g_3	g_3	g_3				$\{g_3\}$
	4	g_0, g_3	g_3	g_0, g_3		g_0		$\{g_0, g_3\}$
\leftarrow	5	g_0, g_2, g_3	g_2, g_3	g_0, g_3		g_0	g_2	$\{g_0, g_2, g_3\}$
	6	g_1, g_3	g_1, g_3	g_3	g_1			$\{g_1, g_3\}$
\leftarrow	7	g_0, g_1, g_2, g_3	g_1, g_2, g_3	g_0, g_3	g_1	g_0	g_2	$\{g_0, g_1, g_2, g_3\}$
	8	g_0, g_1, g_3	g_1, g_3	g_0, g_3	g_1	g_0		$\{g_0, g_1, g_3\}$

The construction of the NFA \mathfrak{N}_{min} is shown in Table 29. For each grid $g = P \times R$, we show its set of quotients P , with $K_i \in P$ replaced by i . For each input $x \in \Sigma$, we give $x^{-1}P$, and then the intersection of the $f(K_i)$ for $K_i \in x^{-1}P$. For example, the set P for g_0 is expressed as $\{0, 4, 5, 7, 8\}$, the set of quotients $a^{-1}P$ of the set P by a is $\{1, 6, 7\}$, and $\eta_C(g_0, a) = f(1) \cap f(6) \cap f(7) = \{g_1\} \cap \{g_1, g_3\} \cap \{g_0, g_1, g_2, g_3\} = \{g_1\}$. Table 26 shows the constructed NFA \mathfrak{N}_{min} , where g_i 's are replaced by i 's. Since \mathfrak{N}_{min} is equivalent to \mathfrak{D} , C is a legal cover. However, \mathfrak{N}_{min} is not atomic, since the right language of state g_2 is not a union of atoms, although it includes atoms A and E as its subsets. The right languages of the other states of \mathfrak{N}_{min} are sets of atoms: $L(g_0) = B \cup D \cup F$, $L(g_1) = C \cup E \cup F$, and $L(g_3) = D \cup E \cup F$.

We believe that NFA's defined by grids are a topic for future research.

6 Conclusions

We have studied the properties of atomic NFA's. We have shown that atoms play an important role in NFA minimization and proved that it is not enough to search for atomic NFA's only.

Table 29. Construction of NFA \mathfrak{N}_{min} .

	g	P	a	a	b	b
			$a^{-1}P$	$\eta_C(g, a)$	$b^{-1}P$	$\eta_C(g, b)$
\rightarrow	g_0	$\{0, 4, 5, 7, 8\}$	$\{1, 6, 7\}$	$\{g_1\}$	$\{2, 7\}$	$\{g_1, g_2\}$
	g_1	$\{1, 2, 6, 7, 8\}$	$\{3, 5, 6, 7\}$	$\{g_3\}$	$\{4, 7, 8\}$	$\{g_0, g_3\}$
\leftarrow	g_2	$\{2, 5, 7\}$	$\{5, 7\}$	$\{g_0, g_2, g_3\}$	$\{2, 4, 7\}$	\emptyset
	g_3	$\{3, 4, 5, 6, 7, 8\}$	$\{3, 6, 7\}$	$\{g_3\}$	$\{1, 2, 7, 8\}$	$\{g_1\}$

References

1. Arnold, A., Dicky, A., Nivat, M.: A note about minimal non-deterministic automata. Bull. EATCS 47, 166–169 (1992)
2. Brzozowski, J.: Canonical regular expressions and minimal state graphs for definite events. In: Proc. Symp. on Mathematical Theory of Automata. MRI Symposia Series, vol. 12, pp. 529–561. Polytechnic Institute of Brooklyn, N.Y. (1963)
3. Brzozowski, J., Tamm, H.: Theory of átomata. In: Mauri, G., Leporati, A. (eds.) DLT 2011. LNCS, vol. 6795, pp. 105–116. Springer (2011)
4. Brzozowski, J., Tamm, H.: Quotient complexities of atoms of regular languages. In: Yen, H.C., Ibarra, O. (eds.) DLT 2012. LNCS, vol. 7410, pp. 50–61. Springer (2012)
5. Ilie, L., Yu, S.: Reducing NFAs by invariant equivalences. Theoret. Comput. Sci. 306, 373–390 (2003)
6. Kameda, T., Weiner, P.: On the state minimization of nondeterministic automata. IEEE Trans. Comput. C-19(7), 617–627 (1970)
7. Matz, O., Potthoff, A.: Computing small finite nondeterministic automata. In: Engberg, U.H., Larsen, K.G., Skou, A. (eds.) Proc. Workshop on Tools and Algorithms for Construction and Analysis of Systems. pp. 74–88. BRICS, Aarhus, Denmark (1995)
8. Ott, G., Feinstein, N.: Design of sequential machines from their regular expressions. J. ACM 8, 585–600 (1961)
9. Polák, L.: Minimalizations of NFA using the universal automaton. Internat. J. Found. Comput. Sci. 16(5), 999–1010 (2005)
10. Rabin, M., Scott, D.: Finite automata and their decision problems. IBM J. Res. and Dev. 3, 114–129 (1959)
11. Sengoku, H.: Minimization of nondeterministic finite automata. Master’s thesis, Kyoto University, Department of Information Science, Kyoto, Japan (1992)